

RANK GRADIENT IN CO-FINAL TOWERS OF CERTAIN KLEINIAN GROUPS

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ABSTRACT. We prove that if the fundamental group of an orientable finite volume hyperbolic 3-manifold has finite index in the reflection group of a right-angled ideal polyhedra in \mathbb{H}^3 then it has a co-final tower of finite sheeted covers with positive rank gradient. The manifolds we provide are also known to have co-final towers of covers with zero rank gradient.

1. INTRODUCTION

Let G be a finitely generated group. The *rank* of G is the minimal cardinality of a generating set, and is denoted by $\text{rk}(G)$. If G_j is a finite index subgroup of G , the Reidemeister-Schreier process ([LS]) gives an upper bound on the rank of G_j .

$$\text{rk}(G_j) - 1 \leq [G : G_j](\text{rk}(G) - 1)$$

Recently Lackenby introduced the notion of *rank gradient* ([La1]). Given a finitely generated group G and a collection $\{G_j\}$ of finite index subgroups, the *rank gradient* of the pair $(G, \{G_j\})$ is defined by

$$\text{rgr}(G, \{G_j\}) = \lim_{j \rightarrow \infty} \frac{\text{rk}(G_j) - 1}{[G : G_j]}$$

We say that the collection of finite index subgroups $\{G_j\}$ is *co-final* if $\bigcap_j G_j = \{1\}$, and we call it a *tower* if $G_{j+1} < G_j$.

In some particular cases it is easy to determine rank gradient, for example:

- (1) When G is a free group, the rank gradient of any pair $(G, \{G_j\})$ is positive.
- (2) The same is true if G is the fundamental group of a closed surface S with $\chi(S) < 0$;
- (3) If $G \twoheadrightarrow F_2$, where F_2 is the free group on two generators then, using (1), one can find a tower (not co-final) of subgroups with positive rank gradient;

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- (4) If G is virtually abelian or if G is the fundamental group of a virtually fibered 3-manifold then there are towers with zero rank gradient. In the latter case we consider the subgroups coming from the cyclic covers of the fibered manifold.
- (5) $\mathrm{SL}(n, \mathbb{Z})$, $n > 2$, has zero rank gradient with respect to towers of congruence subgroups ([Ti], [La1]).

However, determining the rank gradient of a co-final tower is very hard in general. For example, the following question is the motivation for this note:

Question 1. *Does there exist a torsion free finite covolume Kleinian group G with a co-final tower $\{G_j\}$ such that $\mathrm{rgr}(G, \{G_j\}) > 0$.*

The main result of this note provides infinitely many such examples. To state it we introduce some notation.

If M_1 is an orientable finite volume hyperbolic 3-manifold, we call the family of covers $\{M_j \rightarrow M_1\}$ *co-final* (resp. a *tower*) if $\{\pi_1(M_j)\}$ is co-final (resp. a tower). By rank gradient of the pair $(M_1, \{M_j\})$, $\mathrm{rgr}(M_1, \{M_j\})$, we mean the rank gradient of $(\pi_1(M_1), \{\pi_1(M_j)\})$.

Theorem 3.1. *Let M_1 be an orientable finite volume hyperbolic 3-manifold whose fundamental group has finite index in the reflection group of a totally geodesic right-angled ideal polyhedron P_1 in \mathbb{H}^3 . Then there exists a co-final tower of finite sheeted covers $\{M_j \rightarrow M_1\}$ with positive rank gradient.*

This theorem relates to the work of Abért and Nikolov ([AN]), and in particular to a question about *cost of group actions* ([Ga]).

Question 2. *Let G be finitely generated and $\{G_j\}$ be a co-final tower of normal subgroups of G . Does $\mathrm{rgr}(G, \{G_j\})$ depend on the tower $\{G_j\}$?*

Our result provides negative evidence for this question. If one could improve Theorem 3.1 by finding a co-final tower $\{M_j \rightarrow M_1\}$ of regular covers with positive rank gradient, then we claim it would also be possible to find one with zero rank gradient. In fact, Agol proved in [Ag] that if the fundamental group of a 3-dimensional manifold satisfies an algebraic condition, called RFRS, then it virtually fibers. He also proved in [Ag] that the manifolds of the type considered in Theorem 3.1 are virtually RFRS. Therefore, given M_1 as in Theorem 3.1, it is possible to find a tower $\{\Gamma_j\}$ with $\mathrm{rgr}(\pi_1(M_1), \{\Gamma_j\}) = 0$. By taking the *core* of Γ_j in $\pi_1(M_1)$ (i.e., $\mathrm{core}(\Gamma_j) = \bigcap_{g \in \pi_1(M_1)} g\Gamma_j g^{-1}$), one sees that the tower of normal subgroups $\{\mathrm{core}(\Gamma_j)\}$ has zero rank gradient. The desired co-final tower with zero rank gradient would be given by $\{\pi_1(M_j) \cap \mathrm{core}(\Gamma_j)\}$.

The main idea of the proof of Theorem 3.1 is as follows: given P_1 as in the theorem, construct a collection of polyhedra $\{P_j\}$ whose reflection groups have finite index 2^{j-1} in the reflection group of P_1 . If one is given an orientable hyperbolic 3-manifold M_1 whose fundamental group has finite index in the reflection group of P_1 then M_1 has at least as many cusps as the number of vertices of P_1 . We may find manifold covers $M_j \rightarrow M_1$ so that M_j is a 2^{j-1} -sheeted covering and has at least as many cusps as the number of ideal vertices of P_j . We then show that the P_j can be chosen so that the number of its vertices is of the same magnitude as 2^j .

The paper will be organized as follows: section 2 sets up notation and we recall a characterization of right-angled ideal polyhedra using Andreev's theorem ([An]). We then show how the construction of the family $\{P_j\}$ will be done. In section 3 we prove Theorem 3.1. Section 4 contains all the technical results we need to estimate $\text{rk}(\pi_1(M_j))$. In section 5 we show how to construct $\{P_j\}$ so that the family $\{M_j\}$ is co-final. The idea for this appears in [Ag] (Theorem 2.2) and we include a proof here for completeness. Section 6 contains some final remarks and further questions.

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2. SET UP

An *abstract polyhedron* \mathcal{P}_1 is a cell complex on S^2 which can be realized by a convex Euclidean polyhedron. A *labeling* of \mathcal{P}_1 is a map

$$\Theta : \text{Edges}(\mathcal{P}_1) \rightarrow (0, \pi/2]$$

The pair (\mathcal{P}_1, Θ) is a labeled abstract polyhedron. A labeled abstract polyhedron is said to be *realizable* as a hyperbolic polyhedron if there exists a hyperbolic polyhedron P_1 such that there is a label preserving graph isomorphism between the 1-skeleton of P_1 with edges labeled by dihedral angles and the 1-skeleton of \mathcal{P}_1 with edges labeled by Θ .

Let P_1 be a totally geodesic right-angled ideal polyhedron in \mathbb{H}^3 (that is, faces of P_1 are contained in hyperplanes and all vertices of P_1 lie in the boundary at infinity S_∞^2 , where we here we consider the ball model for \mathbb{H}^3). We consider the 1-skeleton of P_1 as a graph $\Gamma_1 \subset S^2$ with labels

$\theta_e = \pi/2$. Let Γ_1^* be its dual graph. A k -circuit is a simple closed curve composed of k edges in Γ_1^* . A *prismatic k -circuit* is a k -circuit γ so that no two edges of Γ_1 which correspond to edges traversed by γ share a vertex. Andreev's theorem for right-angled ideal polyhedra in \mathbb{H}^3 ([An], see also [At]) can be stated as:

Theorem 2.1. *Let \mathcal{P}_1 be an abstract polyhedron. Then \mathcal{P}_1 is realizable as a right-angled ideal polyhedron P_1 if and only if*

- (1) P_1 has at least 6 faces;
- (2) Vertices have valence 4;
- (3) For any triple of faces of P_1 , (f_i, f_j, f_k) , such that $f_i \cap f_j$ and $f_j \cap f_k$ are edges of P_1 with distinct endpoints, $f_i \cap f_k = \emptyset$;
- (4) There are no prismatic 4-circuits.

The above theorem implies that the 1-skeleton of P_1 is a 4-valent graph. The faces can therefore be checkerboard colored. Reflecting P_1 along a face f_1 gives a polyhedron P_2 which is also right-angled, ideal and totally geodesic with checkerboard colored faces (see figure below). We construct a sequence of polyhedra $P_1, P_2, \dots, P_j, \dots$ recursively, whereby P_{j+1} is obtained from P_j by reflection along a face f_j . The faces of P_{j+1} are colored accordingly with the coloring of the faces of P_j .

The notation for the remainder of the paper is as follows: the number of vertices in the face f_j is denoted by S_{f_j} and ϕ_{f_j} denotes the reflection along f_j . B_j and W_j represent the maximal number of ideal vertices on a black or white face of the polyhedron P_j , respectively. V_j denotes the total number of vertices on P_j .

Throughout, the construction of the polyhedra P_j will be done in an alternating fashion with respect to the color of the faces: P_{2j} is obtained from P_{2j-1} by reflection along a black face and P_{2j+1} is obtained from P_{2j} by reflection along a white face.

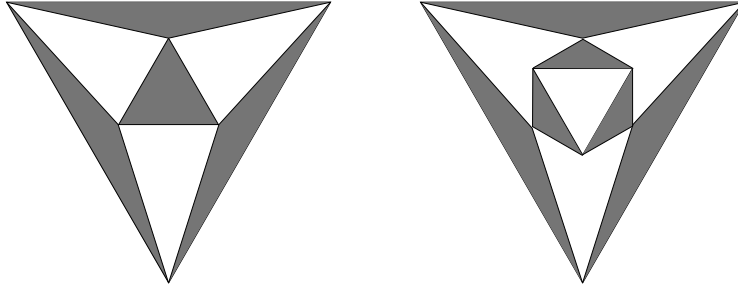


FIGURE 1. Polyhedron P_1 reflected along central black face yields P_2

3. MAIN THEOREM

In this section we prove:

Theorem 3.1. *Let M_1 be an orientable finite volume hyperbolic 3-manifold whose fundamental group has finite index in the reflection group of a right-angled ideal polyhedron P_1 in \mathbb{H}^3 . Then there exists a co-final tower of finite sheeted covers $\{M_j \rightarrow M\}$ with positive rank gradient.*

Our construction of the family $\{M_j\}$ was inspired by the proof of Theorem 2.2 of Agol's paper ([Ag]). The proof that this family can be made co-final is given in section 5 (following [Ag]).

Proof of Theorem 3.1. Consider the family of polyhedra $\{P_j\}$ obtained from P_1 as described above. Denote by G_j the reflection group of P_j and observe that G_{j+1} is a subgroup of G_j of index 2. G_1 acts on \mathbb{H}^3 with fundamental domain P_1 . The orbifold \mathbb{H}^3/G_1 is non-orientable, and may be viewed as P_1 with its faces mirrored. The singular locus is the 2-skeleton of P_1 . Each ideal vertex of P_1 corresponds to a cusp of \mathbb{H}^3/G_1 .

Let M_1 be an orientable cusped hyperbolic 3-manifold such that $\pi_1(M_1)$ has finite index in G_1 . Let $M_j \rightarrow M_1$ be the cover of M_1 whose fundamental group is $\pi_1(M_j) = \pi_1(M_1) \cap G_j$. Since $[G_j : G_{j+1}] = 2$, we must have $[\pi_1(M_j) : \pi_1(M_{j+1})] \leq 2$. Also note that since $\text{vol}(P_j) = 2^{j-1}\text{vol}(P_1)$, for all but finitely many j (at most $[G_1 : \pi_1(M_1)]$) we must have $[\pi_1(M_j) : \pi_1(M_{j+1})] = 2$. We may thus assume that $[\pi_1(M_j) : \pi_1(M_{j+1})] = 2$. By mirroring the faces of P_j , it may be regarded as a non-orientable finite volume orbifold (as described before). This implies that $M_j \rightarrow P_j$ is an orientable finite sheeted cover for $j = 1, 2, \dots$

Note that $[\pi_1(M_1) : \pi_1(M_j)] = 2^{j-1}$. Thus to show that the family $\{M_j \rightarrow M_1\}$ has positive rank gradient we will establish that $\text{rk}(\pi_1(M_j))$ grows with the same magnitude as 2^j .

By “half lives half dies”, an easy lower bound on the rank of the fundamental group of an orientable finite volume hyperbolic 3-manifold is the number of its cusps. Since the cusps of P_j correspond to its ideal vertices and the number of cusps does not go down under finite sheeted covers, it must be that M_j has at least as many cusps as the number of ideal vertices of P_j .

Recall that B_j and W_j are the maximal number of ideal vertices on a black or white face of the polyhedron P_j , respectively, and V_j is the total number of vertices on P_j . The claims below (proved in section 4) gives us the estimates we need for V_j in terms of V_1 , B_1 and W_1 .

Claim 1. $V_1 \geq B_1 + W_1 - 1$

Claim 2. *For any $j \geq 6$,*

$$V_j \geq 2^{j-1}V_1 - 2^{j-1}(B_1 + W_1) + 2^{j-1} + 2^{j-2}$$

Given these, we argue as follows:

$$\begin{aligned} \text{rgr}(M_1, \{M_j\}) &= \lim_{j \rightarrow \infty} \frac{\text{rk}(\pi_1(M_j)) - 1}{[\pi_1(M_1) : \pi_1(M_j)]} \geq \\ \lim_{j \rightarrow \infty} \frac{V_j - 1}{2^{j-1}} &\geq \lim_{j \rightarrow \infty} \frac{2^{j-1}V_1 - 2^{j-1}(B_1 + W_1) + 2^{j-1} + 2^{j-2} - 1}{2^{j-1}} \geq \\ \lim_{j \rightarrow \infty} \frac{2^{j-1}(B_1 + W_1 - 1) - 2^{j-1}(B_1 + W_1) + 2^{j-1} + 2^{j-2} - 1}{2^{j-1}} &\geq \\ \lim_{j \rightarrow \infty} \frac{2^{j-2} - 1}{2^{j-1}} &= \frac{1}{2} \end{aligned}$$

which proves the theorem. \square

4. LOWER BOUNDS ON NUMBER OF IDEAL VERTICES OF P_j

We now proceed to prove Claims 1 and 2. This requires several preliminary results.

Lemma 4.1. *Let P_{j+1} be obtained from P_j by reflection along a face f_j . Then $V_{j+1} = 2V_j - S_{f_j}$.*

Proof. Here we abuse notation and write $v \in f_j$ if v is an ideal vertex of the face f_j and write $v \notin f_j$ otherwise. Note that if $v \notin f_j$, then v yields two vertices on P_{j+1} , namely, v and $\phi_{f_j}(v)$. If $v \in f_j$, then it yields a single vertex (v itself).

If $v \notin f_j$, then, by the observation above, v yields two ideal vertices on P_{j+1} . Since a total of S_{f_j} ideal vertices lie in f_j and $V_j - S_{f_j}$ do not, it must be that that

$$V_{j+1} = 2(V_j - S_{f_j}) + S_{f_j} = 2V_j - S_{f_j}$$

\square

Recall also that the construction of the family of polyhedra $\{P_j\}$ is made in an alternating fashion with respect to the color of the faces: P_{2j} is obtained from P_{2j-1} by reflection along a black face and P_{2j+1} is obtained from P_{2j} by reflection along a white face.

Corollary 4.2. *For $j \geq 1$*

- (1) $V_{2j} \geq 2V_{2j-1} - B_{2j-1}$
- (2) $V_{2j+1} \geq 2V_{2j} - W_{2j}$

Proof. P_{2j} is obtained from P_{2j-1} by reflection along a black face f_{2j-1} , thus $S_{f_{2j-1}} \leq B_{2j-1}$. By the lemma, $V_{2j} = 2V_{2j-1} - S_{f_{2j-1}}$ and therefore $V_{2j} \geq 2V_{2j-1} - B_{2j-1}$. The second inequality is similar. \square

With the notation established above we now find lower bounds for the V_j in terms of V_1, B_1 and W_1 . First we need to find upper bounds for B_j and W_j in terms of B_1 and W_1 . To do this in a way that will fit our purposes we establish two properties of the family $\{P_j\}$. As before, denote by ϕ_{f_j} the reflection along the face f_j .

Lemma 4.3. (1) *If P_j is reflected along a white (resp. black) face f_j , all black faces f_* (resp. white faces f_*) adjacent to f_j yield new black faces \tilde{f}_* (resp. white faces \tilde{f}_*) on P_{j+1} . The number $S_{\tilde{f}_*}$ (resp. $S_{\tilde{f}_*}$) of ideal vertices on \tilde{f}_* (resp. \tilde{f}_*) is $2S_{f_*} - 2$ (resp. $2S_{f_*} - 2$).*

(2) *A face f_* not adjacent to f_j yield two new faces, f_* itself and $\phi_f(f_*)$, both with S_{f_*} vertices.*

Proof. For the first property, reflecting f_* along f_j gives a face $\phi_{f_j}(f_*)$ in P_{j+1} adjacent to f_* . The dihedral angle between f_* and $\phi_f(f_*)$ is π . Thus, on P_{j+1} , they correspond to a single face denoted by \tilde{f}_* . The number of ideal vertices on \tilde{f}_* is exactly $2S_{f_*} - 2$. The second property should be clear. See figure 1 for an illustration of these properties. \square

As an immediate consequence we have

Corollary 4.4.

$$(1) \begin{cases} B_{2j} = B_{2j-1} \\ W_{2j} \leq 2W_{2j-1} - 2 \end{cases}$$

$$(2) \begin{cases} B_{2j+1} \leq 2B_{2j} - 2 \\ W_{2j+1} = W_{2j} \end{cases}$$

We are now in position to estimate the values B_j and W_j in terms of B_1 and W_1 .

Theorem 4.5. *With the notation as before we have*

$$(1) \quad W_{2j+1} = W_{2j} \leq 2^j W_1 - \sum_{l=1}^j 2^l$$

$$(2) \quad B_{2j+2} = B_{2j+1} \leq 2^j B_1 - \sum_{l=1}^j 2^l$$

Proof. We proceed by induction. By corollary 4.4 these statements are true for $j = 1$. Suppose it is also true for $j \leq n$. We now want to

estimate $B_{2n+3} = B_{2n+4}$ and $W_{2n+2} = W_{2n+3}$. The hypothesis is that

$$W_{2j+1} = W_{2j} \leq 2^n W_1 - \sum_{l=1}^n 2^l$$

$$B_{2n+2} = B_{2n+1} \leq 2^n B_1 - \sum_{l=1}^n 2^l$$

P_{2n+2} is obtained from P_{2n+1} by reflection along a black face, denoted by f . White faces on P_{2n+1} adjacent to f yield new white faces on P_{2n+2} with at most $2W_{2n+1} - 2$ vertices, by Corollary 4.4. But

$$2W_{2n+1} - 2 \leq 2[2^n W_1 - \sum_{l=1}^n 2^l] - 2 = 2^{(n+1)} W_1 - \sum_{l=1}^{n+1} 2^l$$

which gives the desired result for W_{2n+2} and W_{2n+3} . Finally, P_{2n+3} is obtained from P_{2n+2} by a reflection along a white face, again denoted by f . Since black faces of P_{2n+2} have at most $B_{2n+2}(= B_{2n+1})$ vertices, black faces of P_{2n+3} will have at most $2B_{2n+1} - 2$ vertices, again by corollary 4.4. But

$$2B_{2n+1} - 2 \leq 2[2^n B_1 - \sum_{l=1}^n 2^l] - 2 = 2^{(n+1)} B_1 - \sum_{l=1}^{n+1} 2^l$$

vertices. This establishes the result for B_{2n+3} and B_{2n+4} . \square

Theorem 4.6. *With the notation as before, and for $j \geq 3$,*

$$\begin{aligned} (1) \quad V_{2j} &\geq 2^{2j-1} V_1 - B_1 \sum_{l=j-1}^{2j-2} 2^l - W_1 \sum_{l=j}^{2j-2} 2^l + \sum_{l=j+2}^{2j-1} 2^l + 2^j + 2 \\ (2) \quad V_{2j+1} &\geq 2^{2j} V_1 - B_1 \sum_{l=j}^{2j-1} 2^l - W_1 \sum_{l=j}^{2j-1} 2^l + \sum_{l=j+2}^{2j} 2^l + 2 \end{aligned}$$

Proof. Lower bounds estimates for V_1, \dots, V_7 are found recursively. V_1, V_2, V_3, V_4 and V_5 do not fit these formulas but V_6 and V_7 do. The statement is then true for $j = 3$. We now proceed by induction, using the previous proposition and corollary 4.2. Suppose it is true for $j \leq n, n \geq 3$. We want to show this implies true for $j = n+1$. By corollary 4.2, $V_{2n+2} \geq 2V_{2n+1} - B_{2n+1}$. The hypothesis is that

$$V_{2n+1} \geq 2^{2n} V_1 - B_1 \sum_{l=n}^{2n-1} 2^l - W_1 \sum_{l=n}^{2n-1} 2^l + \sum_{l=n+2}^{2n} 2^l + 2$$

We also know that

$$B_{2n+1} \leq 2^n B_1 - \sum_{l=1}^n 2^l$$

Thus

$$\begin{aligned} V_{2n+2} &\geq 2V_{2n+1} - B_{2n+1} \geq \\ &2[2^{2n}V_1 - B_1 \sum_{l=n}^{2n-1} 2^l - W_1 \sum_{l=n}^{2n-1} 2^l + \sum_{l=n+2}^{2n} 2^l + 2] - [2^n B_1 - \sum_{l=1}^n 2^l] = \\ &2^{2n+1}V_1 - B_1 \sum_{l=n}^{2n-1} 2^{l+1} - W_1 \sum_{l=n}^{2n-1} 2^{l+1} + \sum_{l=n+2}^{2n} 2^{l+1} + 2^2 + \sum_{l=1}^n 2^l = \\ &2^{2n+1}V_1 - B_1 \sum_{l=n}^{2n} 2^l - W_1 \sum_{l=n+1}^{2n} 2^l + \sum_{l=n+3}^{2n+1} 2^l + 2^{n+1} + 2 \end{aligned}$$

which establishes (1) for $2(n+1) = 2n+2$.

We use the exact same idea to and the estimate for V_{2n+2} to establish (2) for $2(n+1)+1 = 2n+3$. \square

Corollary 4.7. *For any $j \geq 6$,*

$$V_j \geq 2^{j-1}V_1 - 2^{j-1}(B_1 + W_1) + 2^{j-1} + 2^{j-2}$$

Hence Claim 2 in the proof of Theorem 3.1 is proved. We now prove

Claim 1. $V_1 \geq B_1 + W_1 - 1$

Proof. Let f_b and f_w be black and white faces of P_1 with maximal number of vertices, i.e., $S_{f_b} = B_1$ and $S_{f_w} = W_1$.

Case 1: The faces f_b and f_w are not adjacent

Here we get $V_1 \geq B_1 + W_1$ and the claim follows.

Case 2: The faces f_b and f_w are adjacent.

Since f_b and f_w share exactly 2 vertices we see that $V_1 \geq B_1 + W_1 - 2$. Suppose we have equality. Then every vertex of P_1 must be a vertex of either f_b or f_w . Recall that we can visualize the 1-skeleton of P_1 as lying in S^2 . Label the vertices of P_1 by $\{v_1, \dots, v_k\}$. The assumption is that all these vertices lie in the boundary of the disk $D = \overline{(f_b \cup f_w)} \subset S^2$. By Andreev's theorem, P_1 has at least 6 faces, every face is at least 3-sided and all vertices are 4-valent. Denoting by F_1 and E_1 the number of faces and edges of P_1 respectively we have the relation $V_1 - E_1 + F_1 = 2$.

Since vertices are 4-valent we also have $E_1 = 2V_1$. From these relations and $F_1 \geq 6$, we get $V_1 \geq 4$. At two of the vertices, say v_1 and v_2 , three of the emanating edges lie in D and one does not. Denote the ones that do not lie in D by e_1 and e_2 , respectively. At all other v_i we have two edges that lie in D and two that do not. Denote the latter by e_i, e'_i . We have a total of $2(k-2) + 2 = 2k-2$ edges not in D . The problem we have now is combinatorial:

Given the disk $D' = \overline{S^2 - D}$ and the points $v_1, \dots, v_k \in \partial D'$, $k \geq 4$, is it possible to subdivide D' by $2k-2$ edges in a way that exactly one edge emanates from both v_1 and v_2 and exactly two edges emanate from v_3, \dots, v_k in such a way that no pair of edges intersect and every face on the subdivision of D' is at least 3-sided (here we also consider sides coming from the boundary)?

A simple argument will show that the answer to this question is negative. Orient the boundary of D' counterclockwise. Starting at v_1 , draw the edge e_1 emanating from it. The other endpoint of e_1 is some vertex v_{i_1} . Consider the vertices contained in the segment $[v_1, v_{i_1}] \subset \partial D'$ in the given orientation. If there are no vertices at all, then we must have a 2-sided face, which is not possible. Therefore, by relabeling, we may assume v_2 is the first vertex between v_1 and v_{i_1} . Observe that the edges emanating from v_2 are trapped between the edge e_1 and $\partial D'$. Draw an edge e_2 emanating from v_2 with the second endpoint v_{i_2} . It must be that v_{i_2} also lies in $[v_1, v_{i_1}]$, or else we find a pair of intersecting edges. As above, there must be a vertex in the segment $[v_2, v_{i_2}]$. By repeating the above argument eventually we find a 2-sided face, which is not possible. Therefore it must be that $V_1 > B_1 + W_1 - 2$. \square

5. CO-FINALNESS

In this section we provide a way of choosing the black or white faces on the polyhedra P_j along which it is reflected in such a way that the resulting family $\{M_j\}$ of manifolds is cofinal. The main result of this section, Theorem 5.1, appears as part of the proof of Theorem 2.2 of [Ag]. We include a proof here for completeness. To better describe this construction we need to change notation slightly by adding another index.

Start with P_1 and relabel it P_{11} . Reflect along a black face f_{11} obtaining P_{12} . Let $\phi_{f_{11}}$ represent such reflection. Observe that if f is adjacent to f_{11} , then $f \cup \phi_{f_{11}}(f)$ corresponds to a single face on P_{12} . We call f and $\phi_{f_{11}}(f)$ *subfaces* of $f \cup \phi_{f_{11}}(f)$. Next reflect P_{12} along a white face f_{12} , which is also a face of P_{11} or contains a face of P_{11} as

a subface, obtaining P_{13} . We construct a subcollection P_{11}, \dots, P_{1k_1} of polyhedra such that

- (i) If P_{1j} is obtained from $P_{1(j-1)}$ by reflection along a white (black) face then $P_{1(j+1)}$ is obtained from P_{1j} by reflection along a black (white) face.
- (ii) Whenever possible, the face f_{1j} must be a face of P_{11} or contain a face of P_{11} as a subface.
- (iii) No faces of P_{11} are subfaces of P_{1k_1} .

Now set $P_{1k_1} := P_{21}$. Suppose P_{n1} has been constructed. Construct the subcollection of polyhedra P_{n1}, \dots, P_{nk_n} such that

- (i) The reflections were performed in an alternating fashion with respect to the color of the faces;
- (ii) Whenever possible, the face f_{nj} must be a face of P_{n1} or contain a face of P_{n1} as a subface.
- (iii) No faces of P_{n1} are subfaces of P_{nk_n} .

Now set $P_{nk_n} := P_{(n+1)1}$. Inductively we obtain a collection of polyhedra

$$P_{11}, P_{12}, \dots, P_{1k_1} := P_{21}, \dots, P_{2k_2} := P_{31}, \dots, P_{nk_n} := P_{(n+1)1}, \dots$$

satisfying (i), (ii) and (iii) above.

Let G_{ij} be the reflection group of P_{ij} and let M_{ij} be the cover of M_{11} whose fundamental group is $\pi_1(M_{ij}) = \pi_1(M_{11}) \cap G_{ij}$. Co-finalness of the family $\{M_{ij} \rightarrow M_{11}\}$ is an immediate consequence of

Theorem 5.1. *Let G_{ij} be as above. Then $\cap_{ij} G_{ij} = \{1\}$.*

In order to prove this theorem we consider the base point for the fundamental group of each P_{ij} (viewed as orbifolds with their faces mirrored) to be the barycenter x_0 of P_{11} .

Proof of Theorem. Set $R_{ij} = \inf_{\gamma} \{\ell(\gamma)\}$, where γ is an arc with endpoints in faces (possibly edges) of P_{ij} going through x_0 . Note that, by construction, $\lim_{i \rightarrow \infty} R_{ij} = \infty$. For a non-trivial element $g \in G_{11}$ set $R_g = \inf_{[\alpha]=g} \{\ell(\alpha)\}$, where α is a loop in P_{11} based at x_0 and $[\alpha]$ represents its homotopy class. Let α_g be a loop in P_{11} based at x_0 such that $[\alpha_g] = g$ and $\ell(\alpha_g) \leq R_g + 1$.

We claim that for sufficiently large i one cannot have $g \in G_{ij}$. In fact, if α_{ij} is any loop in P_{ij} based at x_0 , then this loop bounces off faces of P_{ij} , yielding an arc γ_{ij} through x_0 . Therefore $\ell(\alpha_{ij}) \geq \ell(\gamma_{ij}) \geq R_{ij}$. Since covering maps preserve length of curves, this implies that if i is large enough no such α_{ij} maps to α_g . Thus it is not possible to find a loop representative for g in P_{ij} . \square

6. FINAL REMARKS

Question 3. *Is it possible, in our setting, to obtain a co-final tower of regular covers $\{M_j \longrightarrow M_1\}$ with positive rank gradient?*

A positive answer to this would be very relevant, as it implies that Question 2 has a negative answer. However, the tower constructed in Theorem 3.1 cannot consist of normal subgroups. To see this we argue as follows: using the main theorem in [Ma] we can find a sequence $\{\gamma_j\}$ of hyperbolic elements, $\gamma_j \in G_j$, whose translation lengths are bounded above by 2.634. Since there exist at most finitely many conjugacy classes of hyperbolic elements of bounded translation length in G_1 , it must be that an infinite subsequence $\{\gamma_{j_k}\}$ lie in the same conjugacy class in G_1 . Let γ be a representative of this class and $g_{j_k} \in G_1$ be such that $\gamma_{j_k} = g_{j_k} \gamma g_{j_k}^{-1}$. If the tower $\{G_j\}$ consists of normal subgroups, then $\gamma \in G_{j_k}$, contradicting the fact that $\{G_{j_k}\}$ is co-final.

Question 3 is relevant also because of the following result (see [AN]):

Theorem (Abért-Nikolov). *Either the Rank vs. Heegaard genus conjecture (see below) is false or Question 2 has a negative solution.*

If an orientable 3-manifold M is closed, a Heegaard splitting of M consists of two handlebodies H_1 and H_2 with their boundaries identified by some orientation preserving homeomorphism. Recall that the genus of, say, ∂H_1 gives an upper bound on the rank of $\pi_1(M)$. If M is not closed, these decompositions are given in terms of compression bodies, again denoted by H_1 and H_2 . In order to obtain useful bounds on the rank of $\pi_1(M)$ we restrict ourselves to those decompositions in which H_1 , for instance, is a handlebody. Note that if this is the case, then the genus of ∂H_1 is again an upper bound for the rank of $\pi_1(M)$. Recall that the *Heegaard genus* of M is the minimal genus of a Heegaard surface. A long standing question in 3-dimensional topology is:

Conjecture. *The rank of an orientable finite volume hyperbolic 3-manifold equals its Heegaard genus.*

Another concept due to Lackenby is that of *Heegaard gradient* ([La2]). Given a orientable 3-manifold M and a family $\{M_j\}$ of finite sheeted covers, we define the Heegaard gradient of $\{M_j \longrightarrow M\}$ by

$$\text{Hgr}(M, \{M_j\}) = \lim_{j \rightarrow \infty} \frac{-\chi(S_j)}{d_j}$$

where d_j is the degree of the cover $M_j \longrightarrow M$ and S_j is a minimal genus Heegaard surface for M_j .

Note that if $\text{rgr}(M, \{M_j\}) > 0$, then $\text{Hgr}(M, \{M_j\}) > 0$. An important conjecture that would follow from the “rank versus Heegaard genus” conjecture is

Conjecture. *Let M be a finite volume hyperbolic 3-manifold and $\{M_i \rightarrow M\}$ a family of finite sheeted covers. Then $\text{rgr}(M, \{M_i\}) > 0$ if and only if $\text{Hgr}(M, \{M_i\}) > 0$*

Our results provide examples for which this is true. In ([La2]) Lackenby showed that if $\pi_1(M)$ is an arithmetic lattice in $\text{PSL}(2, \mathbb{C})$, then M has a co-final family of covers (namely, those arising from congruence subgroups) with positive Heegaard gradient. In [LLR] Long, Lubotzky and Reid generalize this result by proving that every finite volume hyperbolic 3-manifold has a co-final family of finite sheeted regular covers for which the Heegaard gradient is positive. These results were also motivation for this note.

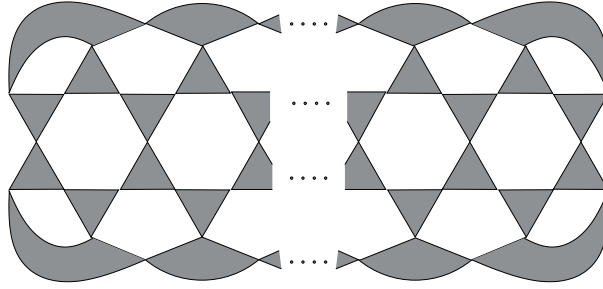
A natural question that arises from our results is to what other categories of finite volume hyperbolic 3-manifolds they hold. For instance:

Question 4. *Is it true that given a right-angled polyhedron P_1 (not necessarily ideal) and a manifold M_1 such that $\pi_1(M_1)$ has finite index in the reflection group of P_1 , then there exists a co-final tower $\{M_j \rightarrow M_1\}$ of finite sheeted covers with positive rank gradient?*

In our setting the ideal vertices played an important role as they were used to find lower bounds on the rank of the fundamental groups. If the polyhedron P_1 has vertices which are not ideal then we need to find another way of estimating the rank of the associated manifolds. Ian Agol has suggested a way for doing this. We are currently working on appropriate bounds for the rank in this case and will include it in a future work.

It is also easy to give examples of families $\{M_j \rightarrow M_1\}$ with arbitrarily large rank gradient. Using the methods above it suffices to provide examples of polyhedra P_1 for which the difference $V_1 - (B_1 + W_1)$ is arbitrarily large. Below we illustrate some cases in which this happens: consider the right-angled ideal polyhedron P_0 pictured below, viewed as lying in S^2 .

Note that, by Andreiev’s theorem, this polyhedron can be realized as a totally geodesic right-angled ideal polyhedron in \mathbb{H}^3 . Reflecting P_0 along the white face containing the point at infinity of S^2 will give us a polyhedron P_1 . Since P_1 is obtained from two copies of P_0 by gluing together the white faces containing the point at infinity, we have a maximum of 6 ideal vertices per white face of P_1 and a maximum of 4 per black faces. Obviously this construction can be made so that P_1

FIGURE 2. Polyhedron P_0

has arbitrarily many ideal vetices. Thus, given any $C > 0$ we may find P_1 such that for the family $\{M_j \rightarrow M_1\}$ as above

$$\lim_{j \rightarrow \infty} \frac{\text{rk}(\pi_1(M_j)) - 1}{[\pi_1(M_1) : \pi_1(M_j)]} \geq \lim_{j \rightarrow \infty} \frac{2^{j-1}(V_1 - (B_1 + W_1)) - 1}{2^{j-1}} > C$$

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